GALILEAN INVARIANCE IN 2+1 DIMENSIONS

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ABSTRACT:
The Galilean invariance in threedimensional space-time is considered. It appears that the Galilei group in 2+1 dimensions posses a three-parameter family of projective representations. Their physical interpretation is discussed in some detail.

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It is well known [1] that, contrary to the case of Poincare group, the Galilei group posses a family of nontrivial projective representations. Moreover, it has been shown [2], [3] that precisely these projective representations are physically meaningful. In particular, with the wave functions transforming according to the true representations of Galilei group one can construct no localized states [2] and no reasonable position operator [3].

The projective representations of $3+1$-dimensional Galilei group were studied in some detail by Levy-Leblond [4]. The family of nonequivalent projective representations is parametrized by one real parameter which is to be identified with the particle mass (the negative mass case calls for some reinterpretation-see Ref. [4]).

Recently, some attention has been paid to the representations of Galilei group in $2+1$ dimensions [5], [6]. It appears that, mainly due to the extremely simple structure of the rotation group in two dimensions, the $2+1$-dimensional Galilei group admits a three-parameter family of nonequivalent projective representations (for a more precise statement see [6]).

In this note we discuss in more detail the possible physical meaning of these projective representations. We conclude that some of them should be simply rejected while the others give rise to quite interesting phenomena. Let us start with a brief description of $2+1$-dimensional Galilei group. It is defined as the group of the space-time transformations of the form

$$\begin{cases} \vec{x}' = \overrightarrow{W(\Theta)x} + \vec{v}t + \vec{u} \\ t' = t + \tau \end{cases} \tag{1}$$

where

$$W(\Theta) = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix} \tag{2}$$

is a rotation. The universal covering group is parametrized by six real parameters $(\Theta, \tau, \vec{v}, \vec{u})$ subject to the following composition law

$$\begin{aligned} &(\Theta', \tau', \vec{v}', \vec{u}') * (\Theta, \tau, \vec{v}, \vec{u}) = \\ &\quad (\Theta' + \Theta, \tau' + \tau, \overrightarrow{W(\Theta')v} + \vec{v}', \overrightarrow{W(\Theta')u} + \tau\vec{v}' + \vec{u}') \end{aligned} \tag{3}$$

The following matrix realisation is often useful

$$\begin{pmatrix} W(\Theta) & \vec{v} & \vec{u} \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \vec{u} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \vec{v} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} W(\Theta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

Accordingly, there exists the following global exponential parametrization

$$g = e^{-i\tau H} e^{i\vec{u}\vec{P}} e^{i\vec{v}\vec{K}} e^{i\Theta J} \tag{5}$$

The corresponding Lie algebra reads

$$\begin{aligned} [J, P_i] &= i\varepsilon_{ij} P_j \\ [J, K_i] &= i\varepsilon_{ij} K_j \\ [K_i, H] &= iP_i \end{aligned} \tag{6}$$

all remaining commutators being vanishing.

In order to find projective representations of Galilei group one has to study the central extensions of the above algebra. This is rather easy. One adds to the right-hand sides of all commutation rules a central element $\mathbf{1}$ with arbitrary coefficients. After (i) making an appropriate redefinitions of P's and K's and (ii) using Jacobi identities one arrives at the following three-parameter family of central extensions

$$\begin{aligned} [P_i, K_j] &= -im\delta_{ij}\mathbf{1} \\ [K_i, K_j] &= ik\varepsilon_{ij}\mathbf{1} \\ [J, H] &= ig\mathbf{1} \\ [K_i, H] &= iP_i \\ [J, P_i] &= i\varepsilon_{ij} P_j \\ [J, K_i] &= i\varepsilon_{ij} K_j, \end{aligned} \tag{7}$$

remaining commutators being vanishing. Let us note some properties of the above algebra:

(i) if $g = 0$ one can obtain the above structure from the contraction of $2+1$-dimensional Poincare algebra. The latter reads

$$
\begin{aligned}
[J, P_i] &= i\varepsilon_{ij} P_j \\
[J, L_i] &= i\varepsilon_{ij} L_j \\
[J, P_0] &= 0 \\
[L_i, L_j] &= -i\varepsilon_{ij} J \\
[L_i, P_j] &= i\delta_{ij} P_0 \\
[L_i, P_0] &= iP_i
\end{aligned}
\tag{8}
$$

Now we add one central element $\mathbf{1}$ to obtain the direct sum of Poincare algebra and onedimensional one and redefine

$$
\begin{aligned}
P_0 &\to mc\mathbf{1} + H/c \\
J &\to -kc^2\mathbf{1} + J \\
L_i &\to cK_i
\end{aligned}
\tag{9}
$$

Letting the contraction parameter $c \to \infty$ we get (7). The case $g \neq 0$ cannot be obtained by contraction from Poincare algebra.
(ii) It is extremely important to observe that, for $m \neq 0$, one can get rid of the parameter k by redefining

$$
K_i \to K_i + \frac{k}{2m}\varepsilon_{ij} P_j; \tag{10}
$$

all remaining commutators are unaffected by this transformation. This remark plays an important role in what follows.
(iii) From the Casimir operators for Poincare group

$$
\begin{aligned}
\tilde{C}_1 &= P_0^2 - \vec{P}^2 \\
\tilde{C}_2 &= P_0 J + \varepsilon_{ij} P_i L_j
\end{aligned}
$$

one obtains by contraction procedure the Casimir operators for the case $g = 0$:

$$C_1 = H - \vec{P}^2/2m \text{ (internal energy)}$$

$$C_2 = J - \frac{1}{m}\vec{K} \times \vec{P} - \frac{kH}{m} \text{ (spin)} \tag{11}$$

The case $g \neq 0$ is slightly more involved. Assume first that $m \neq 0$. We can use eq.(10) to get rid of k. Now, let $C(\vec{K}, \vec{P}, H, J)$ be a Casimir operator. Using (7) we get (with $C_{1,2}$ given by eq.(11) with $k = 0$)

$$\begin{aligned} e^{i\lambda C_1} C(\vec{K}, \vec{P}, H, J) e^{-i\lambda C_1} &= C(\vec{K}, \vec{P}, H, J + \lambda g \mathbf{1}) \\ e^{i\lambda C_2} C(\vec{K}, \vec{P}, H, J) e^{-i\lambda C_2} &= C(\vec{K}, \vec{P}, H - \lambda g \mathbf{1}, J) \end{aligned} \tag{12}$$

Therefore C cannot depend on H and J. Applying the same reasoning with C_i replaced by K_i or P_i we see that C must be a constant. Assume now that $m = 0$; then $\vec{P}^{\,2}$ is the Casimir operator. Moreover, for $k = 0$ (remember that for $m = 0$ we cannot put $k = 0$ by redefining of boosts), $\vec{K} \times \vec{P}$ is also the Casimir operator. It is now easy to find the composition law for projective representations. Keeping in mind that one should put $\mathbf{1} \rightarrow 1$ within representation and using

$$U(g) = e^{-i\tau H} e^{i\vec{u}\cdot\vec{P}} e^{i\vec{v}\cdot\vec{K}} e^{i\Theta J} \tag{13}$$

together with the commutation rules (7) we get

$$\begin{aligned} U(g')U(g) &= \omega(g', g) U(g'g) \\ \omega(g', g) &= e^{ig\tau\Theta'} e^{\frac{-i}{2} m\tau \vec{v}'^2} e^{-im\vec{v}'\cdot\overrightarrow{W(\Theta')u}} e^{\frac{-i}{2} k(\vec{v}' \times \overrightarrow{W(\Theta')v})} \end{aligned} \tag{14}$$

Let us now discuss the physical interpretation of the representations under consideration. First of all one can argue that the case $g \neq 0$ is unphysical. Indeed, the Casimir operators either do not exist (if $m \neq 0$) or do not depend on H (if $m = 0$). Therefore, there exists no counterpart of Schrödinger equation in $\vec{x}$-space; there will be no dynamics. This conclusion is supported by the form of ireducible representations for this case [6]. They either contain any (square integrable) function $f(\vec{p}, \varepsilon)$ (if $m \neq 0$) or are concentrated on submanifold $\vec{p} = const$, again with arbitrary ε-dependence. Translated to $t-$ $\vec{x}$-space it means that the time behaviour of wave functions can be arbitrary. Let us now concentrate on $g = 0$ case. It is easy to find the irreducible representations using standard methods. Assume that $m \neq 0$. The extended Galilei group acts transitively on the paraboloid $\varepsilon - \frac{\vec{p}^2}{2m} = v$, v being internal energy. As in the $3+1$-dimensional case one can argue that putting $v = 0$

does not restrict generality [4]. We choose $\vec{p}=0$ as a standard vector and define an arbitrary eigenvector $|\vec{p},s\rangle$ (s to be specified below) by

$$|\vec{p},s\rangle = B(\vec{p})|\vec{0},s\rangle \tag{15}$$

where

$$B(\vec{p}) \equiv e^{\frac{i}{m}\vec{p}\cdot\vec{K}} \tag{16}$$

is the standard boost. Noting that

$$U^{+}(g)\vec{P}U(g) = \overrightarrow{W(\Theta)P} + m\vec{v}\mathbf{1} \tag{17}$$

and using the substitution rule $\mathbf{1}\to 1$ we write

$$U(g)|\vec{p},s\rangle = B(\vec{p}')[B^{+}(\vec{p}')U(g)B(\vec{p})]|\vec{0},s\rangle \tag{18}$$

where $\vec{p}'=\overrightarrow{W(\Theta)p}+m\vec{v}$. The expression in square bracket is an element of the little group of $\vec{p}=\vec{0}$. But this group is generated by J and $\mathbf{1}(=1)$ and is therefore characterized by one number s (spin). Using eq.(7) we arrive at the following explicit form of the representation:

$$\begin{aligned} &U((\Theta,\tau,\vec{v},\vec{u}))|\vec{p},s\rangle = \\ &\quad e^{i(-\frac{\tau}{2}m\vec{v}^2-\frac{\tau}{2m}\vec{p}^2+m\vec{u}\vec{v}+(\vec{u}-\tau\vec{v})\overrightarrow{W(\Theta)p}-\frac{k}{2m}(\vec{v}\times\overrightarrow{W(\Theta)p})+\Theta s)} \\ &\quad |\overrightarrow{W(\Theta)p}+m\vec{v},s\rangle \end{aligned} \tag{19}$$

The states $|\vec{p},s\rangle$ are normalized in the invariant manner by $\langle\vec{p},s|\vec{p}',s\rangle = \delta^{(2)}(\vec{p}-\vec{p}')$. The wave function $f(\vec{p},s)$ is defined by

$$|f\rangle = \int d^2\vec{p} f(\vec{p},s)|\vec{p},s\rangle$$

$$f(\vec{p},s) = \langle\vec{p},s|f\rangle \tag{20}$$

The action of Galilei group on wave functions can be read off from eqs.(19) and (20). We get:
-translations:

$$f(\vec{p},s)\to e^{-i\tau\frac{\vec{p}^2}{2m}}e^{i\vec{u}\vec{p}}f(\vec{p},s) \tag{21a}$$

-rotations

$$f(\vec{p},s)\to e^{i\Theta s}f(\overrightarrow{W(-\Theta)p},s) \tag{21b}$$

-boosts

$$f(\vec{p}, s) \to e^{-\frac{ik}{2m}(\vec{v}\times\vec{p})} f(\vec{p} - m\vec{v}, s) \tag{21c}$$

Correspondingly, the generators read

$$\begin{gathered} \vec{P} = \vec{p}, \quad H = \frac{\vec{p}^2}{2m} \\ J = -i(\vec{p} \times \vec{\nabla}_p) + s \\ K_i = im\frac{\partial}{\partial p_i} - \frac{k}{2m}\varepsilon_{ij} p_j \end{gathered} \tag{22}$$

As we have noted above, one can always redefine K_i's in such a way that k disappears from the algebra. However, such a transformation provides rather a redefinition of physical observables than a canonical transformation. Therefore, we will also discuss the case $k \neq 0$. Due to the commutation rules $[K_i, H] = iP_i$, $[K_i, P_j] = im\delta_{ij} \cdot \mathbf{1}$ it is tempting to define the position operator as $X_i = \frac{1}{m}K_i$. However, this is wrong. To see this let is invoke the natural requirements the position operator should obey [3]: (i) its expectation value should change by $\vec{a}$ when we go from any state to the state obtained by translation through $\vec{a}$, (ii) it should transform like a vector under rotation and (iii) it should be unchanged by boosts. These conditions are met by the following choice

$$X_i = \frac{1}{m}K_i + \frac{k}{m^2}\varepsilon_{ij} P_j \tag{23}$$

It appears now that the position operators do not commute, their commutator being

$$[X_i, X_j] = -\frac{ik}{m^2}\varepsilon_{ij} \tag{24}$$

Such a situation is not possible in three space dimensions due to the lack of invariant second order antisymmetric tensor.

Eq.(24) has far reaching consequences. It is obvious that the momentum wave function preserves its probability interpretation. Let us, however, consider the Fourier-transformed wave function

$$\tilde{f}_s(\vec{x}, t) = \int d^3\vec{p} f(\vec{p}, s) e^{i(\vec{p}\cdot\vec{x} - \frac{\vec{p}^2}{2m}t)} \tag{25}$$

It obeys standard free Schrödinger equation. The action of X_i operators is easily found to be

$$(X_i\tilde{f}_s)(\vec{x},t) = (x_i - \frac{ik}{2m^2}\varepsilon_{ij}\frac{\partial}{\partial x_j})\tilde{f}_s(\vec{x},t) \tag{26}$$

However, by obvious reasons $\tilde{f}_s(\vec{x},t)$ lacks its standard probability interpretation. Instead, the following uncertainty principle holds

$$\triangle X_1 \cdot \triangle X_2 \geq \frac{|k|}{2m^2} \tag{27}$$

It is not difficult to find the wave functions saturating this inequality. They read

$$f(\vec{p}) = F(\gamma p_1 - ip_2)e^{[u+\frac{k}{2m^2}(\gamma p_1 - ip_2)]p_1} \tag{28}$$

here $\gamma \in \mathbf{R}$, $u \in \mathbf{C}$ and F is arbitrary function chosen in such a way that $f(\vec{p})$ is normalizable.

In the classical limit one finds the hamiltonian theory defined by the following Poisson bracket

$$\{F,G\} = \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial x_i} - \frac{k}{m^2}\varepsilon_{ij}\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j} \tag{29}$$

summation over repeated indices being understood.

The conserved generators (their action being defined by the above Poisson bracket) read

$$\vec{P} = \vec{p}, \qquad H = \frac{\vec{p}^2}{2m}$$

$$J = \vec{x}\times\vec{p} - \frac{k\vec{p}^2}{2m^2}, \qquad K_i = m(x_i - \frac{k}{m^2}\varepsilon_{ij}p_j) - p_i t \tag{30}$$

The second terms in expressions for J and $\vec{K}$ are necessary in order to provide a proper transformation law for $\vec{x}$.

Now the important point is to note that due to the remark (ii) made above the classical and quantum theories defined for $k \neq 0$ can be rephrased in terms of standard ones using the following substitution rules

$$\vec{p} = \vec{p}_s$$

$$x_i = x_{si} + \frac{k}{2m^2}\varepsilon_{ij}p_{sj} \tag{31}$$
$$K_i = K_{si} - \frac{k}{2m}\varepsilon_{ij}p_{sj}$$

where the subscript "s" refers to standard theory. As we have noticed above this does not imply that these theories are equivalent from the physical point of view due to different interpretation of basic observables.

As an example let us consider twodimensional harmonic oscillator (actually, this system is not Galilei-invariant but it may be replaced easily by two particles coupled to each other by harmonic force)

$$H = \frac{\vec{p}^{\,2}}{2m} + \frac{m\omega^2\vec{x}^{\,2}}{2} \tag{32}$$

H commutes with angular momentum J (eq.(30)) so we can look for common eigenvectors

$$H|n,l\rangle = E_n|n,l\rangle, \quad J|n,l\rangle = l|n,l\rangle \tag{33}$$

However, we can consider the equivalent standard theory. Under the substitution (31) I takes its standard form, $J = J_s$ while the hamiltonian reads

$$H = \frac{\vec{p}_s^{\,2}}{2m_s} + \frac{m_s\omega_s^2}{2}\vec{x}_s^{\,2} + \gamma J_s$$
$$\frac{1}{m_s} = \frac{1}{m}(1 + \frac{k^2\omega^2}{4m^2}) \tag{34}$$
$$\frac{\omega_s^2}{\omega^2} = (1 + \frac{k^2\omega^2}{4m^2})$$
$$\gamma = \frac{k\omega^2}{2m}$$

This allows us to find the spectrum of H by considering the spectrum of "standard" harmonic oscillator.

Finally, let us consider the case $m = 0$. The $k \neq 0$ case cannot be now obtained from $k = 0$ one by a simple redefinition. However, both for $k = 0$ and $k \neq 0$, $\vec{P}^{\,2}$ is the Casimir operator. The irreducible representations are therefore constrained to live on circle (or point) $\vec{p}^{\,2} = const$. Consequently, they are not localisable on $\mathbf{R}^2$. Moreover, there is again no constraint on energy (relating it to other "observables") which leads to arbitrary time behaviour of space-time wave functions.

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